

The fundamental gap conjecture: a probabilistic approach via the coupling by reflection

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Abstract

The fundamental gap conjecture asserts that the spectral gap of the Schrödinger operator $-\Delta + V$ with Dirichlet boundary condition on the bounded convex domain $\Omega \subset \mathbb{R}^n$ is greater than $\frac{3\pi^2}{D^2}$, provided that the potential $V : \bar{\Omega} \rightarrow \mathbb{R}$ is convex. Here $D > 0$ is the diameter of Ω . Using analytic methods, Andrews and Clutterbuck proved recently a more general spectral gap comparison result which implies the conjecture. In this work we shall give a simple probabilistic proof via the coupling by reflection of the diffusion processes.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded strictly convex domain with smooth boundary $\partial\Omega$, and $V : \Omega \rightarrow \mathbb{R}$ a potential function. Consider the Schrödinger operator $L = -\Delta + V$ on Ω with Dirichlet boundary condition, where Δ is the standard Laplacian operator on \mathbb{R}^n . The operator L has an increasing sequence of eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$, corresponding to the eigenfunctions $(\phi_i)_{i \geq 0}$ which vanish on the boundary $\partial\Omega$. The eigenfunction $\phi_0 > 0$ and eigenvalue λ_0 are also called the ground state and ground state energy, respectively.

Many authors have conjectured (see [13, 2, 17]) that if V is convex, then the spectral gap of the Schrödinger operator $L = -\Delta + V$ is no less than $\frac{3\pi^2}{D^2}$, i.e.

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}, \quad (1.1)$$

where $D = \text{diam}(\Omega)$ is the diameter of Ω . This conjecture was recently completely solved by Andrews and Clutterbuck in the paper [1] (see [1, Section 1] for a systematic account of the progress in various special cases). In their proof, they introduced the notion of *modulus of convexity* of the potential V . More precisely, a function $\tilde{V} \in C^1([0, D/2])$ is called a modulus of convexity of $V \in C^1(\Omega)$ if for all $x, y \in \Omega$, $x \neq y$, one has

$$\left\langle \nabla V(x) - \nabla V(y), \frac{x - y}{|x - y|} \right\rangle \geq 2\tilde{V}'\left(\frac{|x - y|}{2}\right), \quad (1.2)$$

where $\langle \cdot, \cdot \rangle$ and $|\cdot|$ are respectively the inner product and Euclidean norm of \mathbb{R}^n . Intuitively, we may say that V is “more convex” than \tilde{V} . If the sign is reversed, then \tilde{V} is called the *modulus*

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of concavity for V . We extend \tilde{V} to be an even function on $[-D/2, D/2]$ and consider the one dimensional Schrödinger operator $\tilde{L} = -\frac{d^2}{dt^2} + \tilde{V}$ on the symmetric interval $[-D/2, D/2]$, satisfying the Dirichlet boundary condition. Let $\tilde{\phi}_0$ be the ground state of \tilde{L} . Under the condition (1.2), Andrews and Clutterbuck proved in [1, Theorem 1.5] that $\log \tilde{\phi}_0$ is a modulus of concavity of $\log \phi_0$: for all $x, y \in \Omega$, $x \neq y$,

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \leq 2(\log \tilde{\phi}_0)' \left(\frac{|x-y|}{2} \right). \quad (1.3)$$

The sharp estimate (1.3) enables Andrews and Clutterbuck to prove the gap conjecture (1.1), see [1, Proposition 3.2 and Corollary 1.4] for more details.

Our purpose in the present work is to find a probabilistic proof to the fundamental gap conjecture. The idea of estimating the spectral gap using probabilistic methods (in particular, the coupling method) was developed by Professors M.-F. Chen and F.-Y. Wang in the 1990s, see for instance [6, 7] and Chapters 2 and 3 in the monograph [4]. Professor Hsu gave a short introduction of this method in [9, Section 6.7]. In the first version [8], we were able to give probabilistic proofs to the two main results [1, Theorems 2.1 and 4.1] of Andrews and Clutterbuck's work, except [1, Corollary 4.4] in which they presented a rather technical construction of a sequence $(\psi_k)_{k \geq 1}$ of approximating modulus of log-concavity, satisfying the conditions of [1, Theorem 4.1]. Since the idea and structure of [8] are quite close to Andrews and Clutterbuck's paper [1], we want to find an alternative proof to the gap conjecture (1.1) which is more independent. Unfortunately, we are currently unable to prove the general result that the modulus of convexity (1.2) implies the modulus of log-concavity (1.3) of the ground state. Therefore we restrict ourselves to the special case where V is convex, i.e., $\tilde{V} \equiv 0$ in (1.2), and show that (see also [12, (1.1)])

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \leq -\frac{2\pi}{D} \tan \left(\frac{\pi|x-y|}{2D} \right), \quad \forall x, y \in \Omega, x \neq y. \quad (1.4)$$

Note that $\frac{d}{dt}(\log \cos \frac{\pi t}{D}) = -\frac{\pi}{D} \tan \frac{\pi t}{D}$ and $\tilde{\phi}_0(t) = \cos \frac{\pi t}{D}$ ($t \in [-D/2, D/2]$) is the first Dirichlet eigenfunction of $-\frac{d^2}{dt^2}$ on the interval $[-D/2, D/2]$. Therefore (1.4) is a special case of (1.3). With (1.4) in hand, it is quite easy to deduce the gap conjecture (1.1) through probabilistic arguments. We also mention that the estimate (1.4) improves [3, Theorem 6.1], in which Brascamp and Lieb proved that if V is convex, then ϕ_0 is log-concave.

This paper is organized as follows: in Section 2 we prove that (1.4) holds provided the potential $V : \bar{\Omega} \rightarrow \mathbb{R}$ is convex (see Theorem 2.5), then we show in Section 3 that the fundamental gap conjecture follows from (1.4).

2 Modulus of log-concavity of the ground state

In this section we intend to prove that if V is convex, then the ground state ϕ_0 satisfies (1.4). First we introduce some notations. As in Section 1, Ω is a bounded strictly convex domain in \mathbb{R}^n with smooth boundary $\partial\Omega$. Denote by $\rho_{\partial\Omega} : \bar{\Omega} \rightarrow \mathbb{R}_+$ the distance function to the boundary $\partial\Omega$, and N the unit inward normal vector field on $\partial\Omega$. For $r > 0$, define $\partial_r\Omega = \{x \in \Omega : \rho_{\partial\Omega}(x) \leq r\}$. By [14, Corollary 2.3], there exists $r_0 > 0$ such that $\rho_{\partial\Omega}$ is smooth on $\partial_{r_0}\Omega$. Then for any $x \in \partial_{r_0}\Omega$, there exists a unique $x' \in \partial\Omega$ such that $\rho_{\partial\Omega}(x) = |x - x'|$ and $\nabla \rho_{\partial\Omega}(x) = N(x')$. In particular, $\nabla \rho_{\partial\Omega} = N$ on the boundary $\partial\Omega$.

Given a smooth vector field $b : \Omega \rightarrow \mathbb{R}^n$ and a standard Brownian motion $(B_t)_{t \geq 0}$ on \mathbb{R}^n ; we consider the SDE

$$dX_t = \sqrt{2} dB_t + b(X_t) dt, \quad X_0 = x \in \Omega. \quad (2.1)$$

Lemma 2.1. *Assume that the vector field b satisfies*

$$\liminf_{\rho_{\partial\Omega}(x) \rightarrow 0} \rho_{\partial\Omega}(x) \langle b(x), \nabla \rho_{\partial\Omega}(x) \rangle > 1. \quad (2.2)$$

Then for any $x \in \Omega$, almost surely, $X_t \in \Omega$ for all $t \geq 0$.

Proof. Choose $\Phi \in C^\infty(\bar{\Omega}, \mathbb{R}_+)$ such that $\Phi|_{\partial_{r_0}\Omega} = \rho_{\partial\Omega}|_{\partial_{r_0}\Omega}$. Then there is $c_1 > 0$ such that

$$\Delta\Phi(x) \geq -c_1 \quad \text{for all } x \in \bar{\Omega}. \quad (2.3)$$

By the Itô formula,

$$d\Phi(X_t) = \sqrt{2} \langle \nabla\Phi(X_t), dB_t \rangle + \langle \nabla\Phi(X_t), b(X_t) \rangle dt + \Delta\Phi(X_t) dt.$$

It is enough to study the behavior of X_t near the boundary $\partial\Omega$. When $X_t \in \partial_{r_0}\Omega$, by (2.3), we have (cf. [15, (2.2)])

$$d\rho_{\partial\Omega}(X_t) \geq \sqrt{2} dW_t + \langle \nabla\rho_{\partial\Omega}(X_t), b(X_t) \rangle dt - c_1 dt,$$

where W_t is a one-dimensional Brownian motion. By (2.2), we can find $r_1 \in (0, r_0]$ such that for all $x \in \partial_{r_1}\Omega$,

$$\langle \nabla\rho_{\partial\Omega}(x), b(x) \rangle - c_1 \geq \frac{1}{\rho_{\partial\Omega}(x)}.$$

Thus, if $X_t \in \partial_{r_1}\Omega$, we have

$$d\rho_{\partial\Omega}(X_t) \geq \sqrt{2} dW_t + \frac{1}{\rho_{\partial\Omega}(X_t)} dt.$$

Now we can apply [10, Chap. VI, Theorem 3.1] to conclude that, almost surely, $\rho_{\partial\Omega}(X_t) > 0$ for all $t \geq 0$. \square

Next we collect some properties of the ground state ϕ_0 . Indeed, these properties hold for all C^2 -functions $f : \bar{\Omega} \rightarrow \mathbb{R}_+$ satisfying $f|_{\Omega} > 0$, and for every $x \in \partial\Omega$, $f(x) = 0$ while $\nabla f(x) \neq 0$.

Proposition 2.2. *The ground state ϕ_0 verifies*

- (1) *there exists $0 < \theta_0 \leq \theta_1 < +\infty$ such that for all $x \in \partial\Omega$, $\theta_0 \leq |\nabla\phi_0(x)| \leq \theta_1$;*
- (2) *for every $x \in \partial\Omega$, $\nabla\phi_0(x) = |\nabla\phi_0(x)|N(x)$;*
- (3) $\lim_{\rho_{\partial\Omega}(x) \rightarrow 0} \rho_{\partial\Omega}(x) \langle \nabla \log \phi_0(x), \nabla \rho_{\partial\Omega}(x) \rangle = 1$;
- (4) *there exists $\delta_1 > 0$ and $C_1 > 0$, such that for any $x \in \partial_{\delta_1}\Omega$, $\nabla^2 \log \phi_0(x) \leq -C_1/\rho_{\partial\Omega}(x)$, where $\nabla^2 \log \phi_0$ is the Hessian matrix of $\log \phi_0$.*

Proof. The assertions (1) and (2) are well known, and (3) is an easy consequence of (2) and the mean value theorem. Here we give the proof of the last assertion which is a slight modification of [1, Lemma 4.2].

We take $K \in \mathbb{R}$ such that $|\langle \nabla^2 \phi_0(x) y, y \rangle| \leq K|y|^2$ for all $x \in \bar{\Omega}$ and $y \in \mathbb{R}^n$. For $x_0 \in \partial\Omega$ and z tangent to $\partial\Omega$, it holds

$$\langle \nabla^2 \phi_0(x_0) z, z \rangle = -\mathbb{I}(z, z) \langle \nabla \phi_0(x_0), N(x_0) \rangle = -\mathbb{I}(z, z) |\nabla \phi_0(x_0)|$$

by assertion (2), where \mathbb{I} is the second fundamental form of $\partial\Omega$ at x_0 . The strict convexity of $\partial\Omega$ implies $\mathbb{I}(z, z) \geq \kappa|z|^2$ for some $\kappa > 0$ independent of $x_0 \in \partial\Omega$. The vector field $E = \frac{\nabla \phi_0}{|\nabla \phi_0|}$

is smooth near x_0 , so are the projection $\pi^\perp : y \rightarrow \langle y, E \rangle E$ and the orthogonal projection $\pi = \text{Id} - \pi^\perp$. Thus we have $\langle \nabla^2 \phi_0(x_0) \pi y, \pi y \rangle \leq -\theta_0 \kappa |\pi y|^2$. Therefore, there is $r_0 > 0$ depending on x_0, θ_0 and κ such that for all $x \in \Omega \cap B_{r_0}(x_0)$, it holds

$$\langle \nabla^2 \phi_0(x) \pi y, \pi y \rangle \leq -\frac{\theta_0 \kappa}{2} |\pi y|^2 \quad \text{for all } y \in \mathbb{R}^n; \quad (2.4)$$

$$\frac{1}{2} |\nabla \phi_0(x_0)| \leq |\nabla \phi_0(x)| \leq 2 |\nabla \phi_0(x_0)|. \quad (2.5)$$

Since $\partial\Omega$ is compact, we can find a finite number of balls $\{B_{r_i}(x_i) : x_i \in \partial\Omega, r_i > 0\}_{1 \leq i \leq n_0}$ which have the above properties and cover $\partial\Omega$. Then there exists $\hat{r} > 0$ such that $\partial_{\hat{r}}\Omega \subset \bigcup_{i=1}^{n_0} [\Omega \cap B_{r_i}(x_i)]$. Now for any $x \in \partial_{\hat{r}}\Omega$, there is $i \in \{1, \dots, n_0\}$ such that $x \in \Omega \cap B_{r_i}(x_i)$. Hence for any $y \in \mathbb{R}^n$, we have by (2.4) that

$$\begin{aligned} \langle \nabla^2 \phi_0(x) y, y \rangle &= \langle \nabla^2 \phi_0(x) (\pi y + \pi^\perp y), \pi y + \pi^\perp y \rangle \\ &\leq -\frac{\theta_0 \kappa}{2} |\pi y|^2 + 2K |\pi y| \cdot |\pi^\perp y| + K |\pi^\perp y|^2 \\ &\leq -\frac{\theta_0 \kappa}{4} |\pi y|^2 + \left(K + \frac{4K^2}{\theta_0 \kappa} \right) |\pi^\perp y|^2. \end{aligned}$$

The definition of π and the first inequality in (2.5) lead to

$$\langle \nabla \phi_0(x), y \rangle^2 = |\nabla \phi_0(x)|^2 |\pi^\perp y|^2 \geq \frac{|\nabla \phi_0(x_i)|^2}{4} |\pi^\perp y|^2 \geq \frac{\theta_0 |\nabla \phi_0(x_i)|}{4} |\pi^\perp y|^2.$$

Moreover by (3), there is $\tilde{r} > 0$ such that for all $\bar{x} \in \partial_{\tilde{r}}\Omega$, one has $\phi_0(\bar{x}) \leq 2 |\nabla \phi_0(\bar{x})| \rho_{\partial\Omega}(\bar{x})$. In view of the second inequality in (2.5), we get $\phi_0(x) \leq 4 |\nabla \phi_0(x_i)| \rho_{\partial\Omega}(x)$ if $\rho_{\partial\Omega}(x) \leq \hat{r} \wedge \tilde{r}$. Combining the above estimates, we get

$$\begin{aligned} \langle \nabla^2 \log \phi_0(x) y, y \rangle &= \frac{1}{\phi_0(x)} \left[\langle \nabla^2 \phi_0(x) y, y \rangle - \frac{\langle \nabla \phi_0(x), y \rangle^2}{\phi_0(x)} \right] \\ &\leq \frac{1}{\phi_0(x)} \left[-\frac{\theta_0 \kappa}{4} |\pi y|^2 + \left(K + \frac{4K^2}{\theta_0 \kappa} \right) |\pi^\perp y|^2 - \frac{\theta_0}{16 \rho_{\partial\Omega}(x)} |\pi^\perp y|^2 \right]. \end{aligned}$$

If $\rho_{\partial\Omega}(x) \leq \delta_1 := \hat{r} \wedge \tilde{r} \wedge \frac{\theta_0^2 \kappa}{4(4K\theta_0\kappa + 16K^2 + \theta_0^2 \kappa^2)}$, then

$$\langle \nabla^2 \log \phi_0(x) y, y \rangle \leq \frac{1}{\phi_0(x)} \left[-\frac{\theta_0 \kappa}{4} |y|^2 \right] \leq -\frac{\theta_0 \kappa |y|^2}{16 \theta_1 \rho_{\partial\Omega}(x)}.$$

Therefore the desired result holds with $C_1 = \theta_0 \kappa / 16 \theta_1$. \square

Now we start to give a probabilistic proof to the log-concavity estimate (1.4) of the ground state ϕ_0 when V is convex. It is clear that $\log \phi_0$ satisfies the equation

$$\Delta \log \phi_0 + |\nabla \log \phi_0|^2 = V - \lambda_0.$$

Differentiating this equation leads to

$$\Delta(\nabla \log \phi_0) + 2 \langle \nabla \log \phi_0, \nabla(\nabla \log \phi_0) \rangle = \nabla V, \quad (2.6)$$

or equivalently, in component form,

$$\Delta(\partial_i \log \phi_0) + 2 \langle \nabla \log \phi_0, \nabla(\partial_i \log \phi_0) \rangle = \partial_i V, \quad 1 \leq i \leq n.$$

The equation (2.6) suggests us to consider the following SDE

$$dX_t = \sqrt{2} dB_t + 2\nabla \log \phi_0(X_t) dt, \quad X_0 = x \in \Omega. \quad (2.7)$$

By Proposition 2.2(3), we see that the vector field $2\nabla \log \phi_0$ satisfies the condition of Lemma 2.1. Hence, starting from a point $x \in \Omega$, the solution X_t will not arrive at the boundary $\partial\Omega$. Next we consider the *coupling by reflection* of the process $(X_t)_{t \geq 0}$ which was first introduced by Lindvall and Rogers in [11], and further studies can be found in [5]. Define

$$M(x, y) = I_n - 2 \frac{(x - y)(x - y)^*}{|x - y|^2}, \quad x, y \in \mathbb{R}^n, x \neq y,$$

where I_n is the unit matrix of order n and $(x - y)^*$ is the transpose of the column vector $x - y$. The matrix $M(x, y)$ corresponds to the reflection mapping with respect to the hyperplane perpendicular to the vector $x - y$. Fix some $y \in \Omega$, $y \neq x$, and consider the SDE

$$dY_t = \sqrt{2} M(X_t, Y_t) dB_t + 2\nabla \log \phi_0(Y_t) dt, \quad Y_0 = y \in \Omega. \quad (2.8)$$

Define the stopping times

$$\tau_\eta = \inf\{t > 0 : |X_t - Y_t| = \eta\} \quad \text{and} \quad \sigma_\delta = \inf\{t > 0 : \rho_{\partial\Omega}(X_t) \wedge \rho_{\partial\Omega}(Y_t) = \delta\}$$

for small $\eta, \delta > 0$. As η decreases to 0, τ_η tends to the coupling time $\tau = \inf\{t > 0 : X_t = Y_t\}$; we shall set $Y_t = X_t$ for $t \geq \tau$. Then the process $(Y_t)_{t \geq 0}$ is called the coupling by reflection of $(X_t)_{t \geq 0}$. The stopping time σ_δ is the first time that X_t or Y_t reach the area $\partial_\delta\Omega$. As the function $\log \phi_0$ is smooth with bounded derivatives on $\partial_\delta\Omega$ for any fixed $\delta > 0$, we conclude that, almost surely, $\sigma_\delta < \infty$. Since the two processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ do not arrive at the boundary $\partial\Omega$, it holds $\sigma_\delta \uparrow \infty$ almost surely as $\delta \downarrow 0$.

Now we define some processes:

$$\alpha_t := \nabla \log \phi_0(X_t) - \nabla \log \phi_0(Y_t), \quad \beta_t := \frac{X_t - Y_t}{|X_t - Y_t|}$$

and $F_t := \langle \alpha_t, \beta_t \rangle$. We mention that the process F_t always makes sense, even after the coupling time τ . Indeed, $F_t = 0$ almost surely for $t \geq \tau$. Furthermore,

$$F_0 = \left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x - y}{|x - y|} \right\rangle.$$

We deduce from (2.7) and (2.8) that for $t \leq \tau_\eta \wedge \sigma_\delta$,

$$d(X_t - Y_t) = 2\sqrt{2} \beta_t \beta_t^* dB_t + 2\alpha_t dt, \quad X_0 - Y_0 = x - y \neq 0. \quad (2.9)$$

Lemma 2.3. *Assume that the potential $V : \bar{\Omega} \rightarrow \mathbb{R}$ is convex. We have for $t \leq \tau_\eta \wedge \sigma_\delta$,*

$$dF_t \geq \langle \beta_t, dM_t \rangle, \quad (2.10)$$

where

$$M_t = \sqrt{2} \int_0^t [(\nabla^2 \log \phi_0)(X_s) - (\nabla^2 \log \phi_0)(Y_s) M(X_s, Y_s)] dB_s.$$

Proof. To compute the Itô differential of F_t , we shall apply the Itô formula to $\nabla \log \phi_0(X_t)$. We have

$$\begin{aligned} d[\nabla \log \phi_0(X_t)] &= \sqrt{2} (\nabla^2 \log \phi_0)(X_t) dB_t + [2\langle \nabla \log \phi_0, \nabla^2 \log \phi_0 \rangle + \Delta(\nabla \log \phi_0)](X_t) dt \\ &= \sqrt{2} (\nabla^2 \log \phi_0)(X_t) dB_t + \nabla V(X_t) dt, \end{aligned}$$

where the last equality is due to (2.6). In the same way, for $t \leq \tau_\eta \wedge \sigma_\delta$,

$$d[\nabla \log \phi_0(Y_t)] = \sqrt{2} (\nabla^2 \log \phi_0)(Y_t) M(X_t, Y_t) dB_t + \nabla V(Y_t) dt.$$

Now we obtain

$$d\alpha_t = dM_t + (\nabla V(X_t) - \nabla V(Y_t)) dt, \quad t \leq \tau_\eta \wedge \sigma_\delta, \quad (2.11)$$

where M_t is a vector-valued square integrable martingale before the stopping time $\tau_\eta \wedge \sigma_\delta$.

It remains to compute $d\beta_t$. For $t \leq \tau_\eta \wedge \sigma_\delta$, the Itô formula yields

$$\begin{aligned} d\beta_t &= |X_t - Y_t|^{-1} d(X_t - Y_t) + (X_t - Y_t) d(|X_t - Y_t|^{-1}) \\ &\quad + d(X_t - Y_t) \cdot d(|X_t - Y_t|^{-1}). \end{aligned} \quad (2.12)$$

By (2.9), we have

$$d|X_t - Y_t| = \langle \beta_t, 2\sqrt{2} \beta_t \beta_t^* dB_t \rangle + 2\langle \beta_t, \alpha_t \rangle dt = 2\sqrt{2} \langle \beta_t, dB_t \rangle + 2F_t dt, \quad (2.13)$$

where the last equality follows from $|\beta_t| \equiv 1$. Again by the Itô formula,

$$\begin{aligned} d(|X_t - Y_t|^{-1}) &= -\frac{d|X_t - Y_t|}{|X_t - Y_t|^2} + \frac{d|X_t - Y_t| \cdot d|X_t - Y_t|}{|X_t - Y_t|^3} \\ &= -\frac{2\sqrt{2} \langle \beta_t, dB_t \rangle}{|X_t - Y_t|^2} - \frac{2\langle \beta_t, \alpha_t \rangle}{|X_t - Y_t|^2} dt + \frac{8}{|X_t - Y_t|^3} dt. \end{aligned}$$

Combining this identity with (2.9), we get

$$d(X_t - Y_t) \cdot d(|X_t - Y_t|^{-1}) = -\frac{8(X_t - Y_t)}{|X_t - Y_t|^3} dt.$$

Substituting these computations into (2.12), we arrive at

$$d\beta_t = \frac{2}{|X_t - Y_t|} (\alpha_t - \langle \beta_t, \alpha_t \rangle \beta_t) dt, \quad t \leq \tau_\eta \wedge \sigma_\delta. \quad (2.14)$$

Notice that β_t has no martingale part.

Now by the definition of F_t and (2.11), (2.14), we have for $t \leq \tau_\eta \wedge \sigma_\delta$,

$$\begin{aligned} dF_t &= \langle \beta_t, d\alpha_t \rangle + \langle \alpha_t, d\beta_t \rangle + \langle d\alpha_t, d\beta_t \rangle \\ &= \langle \beta_t, dM_t \rangle + \langle \nabla V(X_t) - \nabla V(Y_t), \beta_t \rangle dt + 2 \frac{|\alpha_t|^2 - \langle \alpha_t, \beta_t \rangle^2}{|X_t - Y_t|} dt. \end{aligned}$$

Noticing that the last term is nonnegative and V is convex, we complete the proof. \square

Let $\tilde{\phi}_{D,0}(z) = \cos \frac{\pi z}{D}$, $z \in [-D/2, D/2]$ be the first Dirichlet eigenfunction of the operator $-\frac{d^2}{dz^2}$ on the interval $[-D/2, D/2]$. Here and below we write $\tilde{\phi}_{D,0}$ instead of $\tilde{\phi}_0$ to stress the dependence on the length of the interval $[-D/2, D/2]$. For simplification of notations, set $\psi_D(z) = (\log \tilde{\phi}_{D,0})'(z) = -\frac{\pi}{D} \tan \frac{\pi z}{D}$ which is well defined on $(-D/2, D/2)$. Note that ψ_D makes no sense at $z = \pm D/2$, thus we first take $D_1 > D$ and consider $\tilde{\phi}_{D_1,0}$ and ψ_{D_1} . Then ψ_{D_1} is smooth on $[0, D/2]$ with bounded derivatives and it satisfies

$$\psi_{D_1}'' + 2\psi_{D_1}\psi_{D_1}' = 0. \quad (2.15)$$

Lemma 2.4. Set $\xi_t = |X_t - Y_t|/2$. We have for $t \leq \tau_\eta \wedge \sigma_\delta$,

$$d\psi_{D_1}(\xi_t) = \sqrt{2} \psi'_{D_1}(\xi_t) \langle \beta_t, dB_t \rangle + \psi'_{D_1}(\xi_t) [F_t - 2\psi_{D_1}(\xi_t)] dt.$$

Proof. By (2.13), ξ_t satisfies

$$d\xi_t = \sqrt{2} \langle \beta_t, dB_t \rangle + F_t dt, \quad \xi_0 = |x - y|/2 > 0. \quad (2.16)$$

When $t \leq \tau_\eta \wedge \sigma_\delta$, by (2.16) and the Itô formula,

$$\begin{aligned} d\psi_{D_1}(\xi_t) &= \psi'_{D_1}(\xi_t) [\sqrt{2} \langle \beta_t, dB_t \rangle + F_t dt] + \psi''_{D_1}(\xi_t) dt \\ &= \sqrt{2} \psi'_{D_1}(\xi_t) \langle \beta_t, dB_t \rangle + \psi'_{D_1}(\xi_t) [F_t - 2\psi_{D_1}(\xi_t)] dt, \end{aligned}$$

where the second equality follows from (2.15). \square

Now we are ready to prove

Theorem 2.5 (Modulus of log-concavity). *Assume that the potential function $V : \Omega \rightarrow \mathbb{R}$ is convex. Then for all $x, y \in \Omega$ with $x \neq y$, it holds*

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x - y}{|x - y|} \right\rangle \leq -\frac{2\pi}{D} \tan \left(\frac{\pi|x - y|}{2D} \right). \quad (2.17)$$

Proof. Fix $\eta > 0$, $\delta > 0$ small enough and $D_1 > D$. Combining Lemmas 2.3 and 2.4, we get for $t \leq \tau_\eta \wedge \sigma_\delta$,

$$d[F_t - 2\psi_{D_1}(\xi_t)] \geq d\tilde{M}_t - 2\psi'_{D_1}(\xi_t) [F_t - 2\psi_{D_1}(\xi_t)] dt, \quad (2.18)$$

in which $d\tilde{M}_t = \langle \beta_t, dM_t \rangle - 2\sqrt{2} \psi'_{D_1}(\xi_t) \langle \beta_t, dB_t \rangle$ is the martingale part. The above stochastic differential inequality is the key ingredient to the proof.

The inequality (2.18) is equivalent to

$$d\left([F_t - 2\psi_{D_1}(\xi_t)] e^{\int_0^t 2\psi'_{D_1}(\xi_s) ds}\right) \geq e^{\int_0^t 2\psi'_{D_1}(\xi_s) ds} d\tilde{M}_t.$$

Integrating from 0 to $t \wedge \tau_\eta \wedge \sigma_\delta$ leads to

$$\begin{aligned} &[F_{t \wedge \tau_\eta \wedge \sigma_\delta} - 2\psi_{D_1}(\xi_{t \wedge \tau_\eta \wedge \sigma_\delta})] e^{\int_0^{t \wedge \tau_\eta \wedge \sigma_\delta} 2\psi'_{D_1}(\xi_s) ds} \\ &\geq [F_0 - 2\psi_{D_1}(\xi_0)] + \int_0^{t \wedge \tau_\eta \wedge \sigma_\delta} e^{\int_0^s 2\psi'_{D_1}(\xi_r) dr} d\tilde{M}_s. \end{aligned}$$

Taking expectation on both sides, we obtain

$$F_0 - 2\psi_{D_1}(\xi_0) \leq \mathbb{E}\left([F_{t \wedge \tau_\eta \wedge \sigma_\delta} - 2\psi_{D_1}(\xi_{t \wedge \tau_\eta \wedge \sigma_\delta})] e^{\int_0^{t \wedge \tau_\eta \wedge \sigma_\delta} 2\psi'_{D_1}(\xi_s) ds}\right).$$

Since $\psi'_{D_1}(z) = -\frac{\pi^2}{D_1^2} \sec^2(\frac{\pi z}{D_1})$ is negative for $z \in [0, D/2]$, the term $e^{\int_0^{t \wedge \tau_\eta \wedge \sigma_\delta} 2\psi'_{D_1}(\xi_s) ds} \leq 1$ for all $t > 0$. Note that ψ_{D_1} is a bounded function on $[0, D/2]$, and by Proposition 2.2(4), there exists $K_1 < +\infty$ such that $\nabla^2 \log \phi_0 \leq K_1$, which implies, almost surely, $F_{t \wedge \tau_\eta \wedge \sigma_\delta} \leq K_1 D$ for all $t > 0$. Therefore by the Lebesgue-Fatou Lemma (cf. [18, p.17]), taking the upper limit as $t \rightarrow +\infty$ yields

$$F_0 - 2\psi_{D_1}(\xi_0) \leq \mathbb{E}\left([F_{\tau_\eta \wedge \sigma_\delta} - 2\psi_{D_1}(\xi_{\tau_\eta \wedge \sigma_\delta})] e^{\int_0^{\tau_\eta \wedge \sigma_\delta} 2\psi'_{D_1}(\xi_s) ds}\right). \quad (2.19)$$

Remark 2.6. Recall that Brascamb and Lieb proved in [3, Theorem 6.1] that if the bounded domain Ω and the potential $V : \Omega \rightarrow \mathbb{R}$ are convex, then the ground state ϕ_0 is log-concave. Using this result, we can give a short proof to (2.17). Indeed, by the definition of the process F_t , the log-concavity of ϕ_0 implies that the random variables $F_{\tau_\eta \wedge \sigma_\delta} \leq 0$ almost surely. Hence by (2.19),

$$F_0 - 2\psi_{D_1}(\xi_0) \leq -2\mathbb{E}\left(\psi_{D_1}(\xi_{\tau_\eta \wedge \sigma_\delta})e^{\int_0^{\tau_\eta \wedge \sigma_\delta} 2\psi'_{D_1}(\xi_s) ds}\right).$$

Furthermore, we deduce from [11, Example 5] and the log-concavity of ϕ_0 that the coupling process $(X_t, Y_t)_{t \geq 0}$ is successful, that is, $\tau < +\infty$ almost surely. Note also that ψ_{D_1} is bounded on the interval $[0, D/2]$ and $\psi_{D_1}(0) = 0$. By the dominated convergence theorem, letting η and δ decrease to 0 leads to

$$F_0 - 2\psi_{D_1}(\xi_0) \leq -2\mathbb{E}\left(\psi_{D_1}(\xi_\tau)e^{\int_0^\tau 2\psi'_{D_1}(\xi_s) ds}\right) = 0.$$

Thus we obtain (2.17) with D being replaced by D_1 . Letting $D_1 \rightarrow D$ completes the proof. In the following, however, we would like to give another proof without using Brascamb and Lieb's log-concavity estimate. This proof may be extended to more general cases where (1.2) is verified, provided that one can prove the integrability of the random variables $e^{\int_0^{\tau_\eta \wedge \sigma_\delta} 2\psi'_{D_1}(\xi_s) ds}$, $\eta, \delta > 0$.

Next we show that for any $\varepsilon > 0$, the right hand side of (2.19) is less than ε when η and δ are sufficiently small. To this end we need the following two estimates on the ground state ϕ_0 : the first one concerns the near diagonal behavior of $\nabla \log \phi_0$ while the second one is the asymptotics of $\nabla \log \phi_0$ near the boundary $\partial\Omega$.

Lemma 2.7. For any $\varepsilon > 0$, there is $\eta_1 > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \eta_1$, it holds

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x - y}{|x - y|} \right\rangle \leq \varepsilon.$$

Proof. Recall Proposition 2.2(4) for the choice of $\delta_1 > 0$. We take sufficiently small $\delta_2 \in (0, \delta_1]$, such that for any $x_0, x_1 \in \partial_{\delta_2}\Omega$ with $|x_0 - x_1| \leq \delta_2/2$, the line segment $\ell_s = (1 - s)x_0 + sx_1$ joining x_0 to x_1 lies in $\partial_{\delta_1}\Omega$. Now choose any $x, y \in \Omega$ with $|x - y| \leq \delta_2/2$. We analyze two cases:

(i) If both x and y belong to $\partial_{\delta_2}\Omega$, then by Proposition 2.2(4), we have

$$\begin{aligned} & \left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x - y}{|x - y|} \right\rangle \\ &= \int_0^1 \left\langle [\nabla^2 \log \phi_0((1 - s)y + sx)](x - y), \frac{x - y}{|x - y|} \right\rangle ds \leq 0. \end{aligned}$$

(ii) If one of the two points, say $x \in \Omega \setminus \partial_{\delta_2}\Omega$, then

$$\rho_{\partial\Omega}(y) \geq \rho_{\partial\Omega}(x) - |x - y| > \delta_2 - \delta_2/2 = \delta_2/2.$$

Therefore both x and y belong to $\Omega \setminus \partial_{\delta_2/2}\Omega$. Note that the function $\log \phi_0$ is smooth on the domain $\Omega \setminus \partial_{\delta_2/2}\Omega$ with bounded derivatives of all orders. Hence we can find $\eta_1 \leq \delta_2/2$ such that if $|x - y| \leq \eta_1$, it holds

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x - y}{|x - y|} \right\rangle \leq \varepsilon.$$

The proof is complete. \square

The next result is weaker than [16, Lemma 2.8], but it is sufficient for our purpose.

Lemma 2.8. *Let $\eta_1 > 0$ be given as in Lemma 2.7. There is $\delta_3 > 0$ small enough such that if $\delta < \delta_3$ and $x \in \partial_\delta \Omega$, $y \in \Omega$ with $|x - y| > \eta_1$, it holds*

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x - y}{|x - y|} \right\rangle \leq -C_2 \log \frac{\delta_3}{\delta} + C_3$$

for some constants $C_2, C_3 > 0$.

Proof. The following argument is a slight modification of [16, Lemma 2.8]. By the strict convexity of Ω , we choose $\delta_3 \ll \eta_1$ so that if $x, y \in \Omega$ with $|x - y| > \eta_1$, then the line segment ℓ linking x and y intersects $\Omega \setminus \partial_{\delta_3} \Omega$. Let $\delta < \delta_3$ and $x \in \partial_\delta \Omega$, $y \in \Omega$ with $|x - y| > \eta_1$. Let $\ell(s) = x - s(x - y)/|x - y|$ be the line segment joining x to y , parameterized by arc length. Divide $\ell(s)$ into disjoint curves: ℓ_1 lying in $\partial_{\delta_3} \Omega$ and ℓ_2 lying in $\Omega \setminus \partial_{\delta_3} \Omega$. Then

$$\begin{aligned} \left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x - y}{|x - y|} \right\rangle &= \left\langle \nabla \log \phi_0(\ell(s)), \ell'(s) \right\rangle \Big|_{s=0}^{|x-y|} \\ &= \int_0^{|x-y|} \langle [\nabla^2 \log \phi_0(\ell(s))] \ell'(s), \ell'(s) \rangle ds \\ &=: I_1 + I_2, \end{aligned}$$

where $I_j = \int_{\ell_j} \langle [\nabla^2 \log \phi_0(\ell(s))] \ell'(s), \ell'(s) \rangle ds$, $j = 1, 2$.

From Proposition 2.2(4), we know that there is $K_1 > 0$ such that $\nabla^2 \log \phi_0 \leq K_1$. Therefore

$$I_2 = \int_{\ell_2} \langle [\nabla^2 \log \phi_0(\ell(s))] \ell'(s), \ell'(s) \rangle ds \leq K_1 |x - y| \leq K_1 D. \quad (2.20)$$

Next we estimate I_1 . Remark that if $\rho_{\partial\Omega}(y) < \delta_3$, then ℓ_1 consists of two disjoint line segments separated by ℓ_2 . However, since $\nabla^2 \phi_0$ is negative in $\partial_{\delta_3} \Omega$, we only need to consider the line segment having x as one of the end point. There is a unique integer $k \in \mathbb{Z}_+$ satisfying $e^k \delta \leq \delta_3 < e^{k+1} \delta$. Let s_j be the first s such that $\rho_{\partial\Omega}(\ell(s)) = e^j \delta$, $j = 0, 1, \dots, k$, and \bar{s} the first s satisfying $\rho_{\partial\Omega}(\ell(s)) = \delta_3$. The strict convexity of Ω implies that $s \rightarrow \rho_{\partial\Omega}(\ell(s))$ is increasing for $s \in [0, \bar{s}]$, and it is clear that

$$s_{j+1} - s_j \geq e^{j+1} \delta - e^j \delta = (e - 1) e^j \delta, \quad 0 \leq j \leq k - 1. \quad (2.21)$$

Denote by $\ell[s, t]$ the line segment between $\ell(s)$ and $\ell(t)$ for $0 \leq s \leq t \leq |x - y|$. Then

$$\begin{aligned} I_1 &= \int_{\ell_1} \langle [\nabla^2 \log \phi_0(\ell(s))] \ell'(s), \ell'(s) \rangle ds \\ &\leq \left(\int_{\ell[0, s_0]} + \sum_{j=0}^{k-1} \int_{\ell[s_j, s_{j+1}]} + \int_{\ell[s_k, \bar{s}]} \right) \langle [\nabla^2 \log \phi_0(\ell(s))] \ell'(s), \ell'(s) \rangle ds \\ &\leq \sum_{j=0}^{k-1} \int_{\ell[s_j, s_{j+1}]} \langle [\nabla^2 \log \phi_0(\ell(s))] \ell'(s), \ell'(s) \rangle ds, \end{aligned}$$

where both inequalities follow from $\nabla^2 \log \phi_0 \leq 0$ on $\partial_{\delta_3} \Omega$. By Proposition 2.2(4), we have

$$I_1 \leq \sum_{j=0}^{k-1} \int_{\ell[s_j, s_{j+1}]} \frac{-C_1}{\rho_{\partial\Omega}(\ell(s))} ds.$$

Noting that by (2.21),

$$\int_{\ell[s_j, s_{j+1}]} \frac{ds}{\rho_{\partial\Omega}(\ell(s))} \geq \frac{1}{\rho_{\partial\Omega}(\ell(s_{j+1}))} \int_{\ell[s_j, s_{j+1}]} ds = \frac{s_{j+1} - s_j}{e^{j+1}\delta} \geq \frac{e-1}{e},$$

hence

$$I_1 \leq -C_1 \sum_{j=0}^{k-1} \frac{e-1}{e} = -C_1 \frac{e-1}{e} k \leq -C_1 \frac{e-1}{e} \left(\log \frac{\delta_3}{\delta} - 1 \right), \quad (2.22)$$

where the last inequality is due to the choice of the integer k . Combining (2.20) and (2.22), we finish the proof with $C_2 = C_1(e-1)/e$ and $C_3 = C_2 + K_1 D$. \square

Now we continue the proof of Theorem 2.5. Fix $\varepsilon > 0$. In Lemmas 2.7 and 2.8, we choose η_1 smaller (thus δ_3 should also be chosen again) so that for all $z \in [0, \eta_1]$, it holds $-2\psi_{D_1}(z/2) \leq \varepsilon$, which is possible since ψ_{D_1} is uniformly continuous on $[0, D/2]$ and $\psi_{D_1}(0) = 0$. Moreover, there exists $\delta_4 < \delta_3$ such that $-2\psi_{D_1}(z) \leq C_2 \log \frac{\delta_3}{\delta_4} - C_3$ for all $z \in [0, D/2]$. We have by (2.19),

$$F_0 - 2\psi_{D_1}(\xi_0) \leq \mathbb{E} \left(\left[F_{\tau_{\eta_1} \wedge \sigma_{\delta_4}} - 2\psi_{D_1}(\xi_{\tau_{\eta_1} \wedge \sigma_{\delta_4}}) \right] e^{\int_0^{\tau_{\eta_1} \wedge \sigma_{\delta_4}} 2\psi'_{D_1}(\xi_s) ds} \right). \quad (2.23)$$

On the event $\{\tau_{\eta_1} \leq \sigma_{\delta_4}\}$, we have by Lemma 2.7 and the choice of η_1 that

$$F_{\tau_{\eta_1} \wedge \sigma_{\delta_4}} - 2\psi_{D_1}(\xi_{\tau_{\eta_1} \wedge \sigma_{\delta_4}}) = F_{\tau_{\eta_1}} - 2\psi_{D_1}(\xi_{\tau_{\eta_1}}) \leq 2\varepsilon;$$

while on the event $\{\sigma_{\delta_4} < \tau_{\eta_1}\}$,

$$2\xi_{\tau_{\eta_1} \wedge \sigma_{\delta_4}} = |X_{\sigma_{\delta_4}} - Y_{\sigma_{\delta_4}}| > \eta_1,$$

hence Lemma 2.8 implies

$$F_{\tau_{\eta_1} \wedge \sigma_{\delta_4}} - 2\psi_{D_1}(\xi_{\tau_{\eta_1} \wedge \sigma_{\delta_4}}) = F_{\sigma_{\delta_4}} - 2\psi_{D_1}(\xi_{\sigma_{\delta_4}}) \leq 0.$$

Therefore (2.23) gives us $F_0 - 2\psi_{D_1}(\xi_0) \leq 2\varepsilon$. As $\varepsilon > 0$ is arbitrary, we get $F_0 \leq 2\psi_{D_1}(\xi_0)$, that is to say,

$$\left\langle \nabla \log \phi_0(x) - \nabla \log \phi_0(y), \frac{x-y}{|x-y|} \right\rangle \leq -\frac{2\pi}{D_1} \tan \left(\frac{\pi|x-y|}{2D_1} \right).$$

Letting D_1 decrease to D completes the proof. \square

3 Proof of the gap conjecture

In this section we present a probabilistic proof of the fundamental gap conjecture (1.1), based on the log-concavity estimate (1.4) of the ground state ϕ_0 . We shall make use of the following well-known result (see [1, Proposition 3.1] for a proof).

Lemma 3.1. *Let u_1 be any smooth solution of*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u - Vu \quad \text{on } \mathbb{R}_+ \times \Omega; \\ u &= 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{aligned} \quad (3.1)$$

and u_0 the solution of (3.1) with initial data ϕ_0 , where ϕ_0 is the ground state of the Schrödinger operator $L = -\Delta + V$. Set $v = \frac{u_1}{u_0}$. Then v is smooth on $\mathbb{R}_+ \times \bar{\Omega}$ and satisfies the heat equation

$$\frac{\partial v}{\partial t} = \Delta v + 2\langle \nabla \phi_0, \nabla v \rangle \quad \text{on } \mathbb{R}_+ \times \Omega. \quad (3.2)$$

Recall the processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ defined in Section 2. We still denote by $\xi_t = |X_t - Y_t|/2$ which satisfies (2.16).

Proposition 3.2. *We have for all $t \geq 0$,*

$$\mathbb{E} \sin \left(\frac{\pi \xi_t}{D} \right) \leq \exp \left(- \frac{3\pi^2 t}{D^2} \right) \sin \left(\frac{\pi |x - y|}{2D} \right).$$

Proof. By the Itô formula, we have for $t \leq \tau_\eta \wedge \sigma_\delta$,

$$d \sin \left(\frac{\pi \xi_t}{D} \right) = \frac{\pi}{D} \cos \left(\frac{\pi \xi_t}{D} \right) d\xi_t - \frac{\pi^2}{2D^2} \sin \left(\frac{\pi \xi_t}{D} \right) d\xi_t \cdot d\xi_t.$$

Note that $0 \leq \xi_t < D/2$ almost surely for all $t \geq 0$, hence $\cos \left(\frac{\pi \xi_t}{D} \right) > 0$. The equation (2.16) and the log-concavity estimate (1.4) lead to

$$d\xi_t \leq \sqrt{2} \langle \beta_t, dB_t \rangle - \frac{2\pi}{D} \tan \left(\frac{\pi \xi_t}{D} \right) dt.$$

Therefore

$$\begin{aligned} d \sin \left(\frac{\pi \xi_t}{D} \right) &\leq \frac{\pi}{D} \cos \left(\frac{\pi \xi_t}{D} \right) \left[\sqrt{2} \langle \beta_t, dB_t \rangle - \frac{2\pi}{D} \tan \left(\frac{\pi \xi_t}{D} \right) dt \right] - \frac{\pi^2}{D^2} \sin \left(\frac{\pi \xi_t}{D} \right) dt \\ &= \sqrt{2} \frac{\pi}{D} \cos \left(\frac{\pi \xi_t}{D} \right) \langle \beta_t, dB_t \rangle - \frac{3\pi^2}{D^2} \sin \left(\frac{\pi \xi_t}{D} \right) dt. \end{aligned}$$

To simplify the notations, we denote by \hat{M}_t the martingale part on the right hand side. Then we obtain for $t \leq \tau_\eta \wedge \sigma_\delta$ that

$$d \left[\exp \left(\frac{3\pi^2 t}{D^2} \right) \sin \left(\frac{\pi \xi_t}{D} \right) \right] \leq \exp \left(\frac{3\pi^2 t}{D^2} \right) d\hat{M}_t.$$

Integrating both sides from 0 to $t \wedge \tau_\eta \wedge \sigma_\delta$ and taking expectation yield

$$\mathbb{E} \left[\exp \left(\frac{3\pi^2 (t \wedge \tau_\eta \wedge \sigma_\delta)}{D^2} \right) \sin \left(\frac{\pi \xi_{t \wedge \tau_\eta \wedge \sigma_\delta}}{D} \right) \right] \leq \sin \left(\frac{\pi |x - y|}{2D} \right).$$

Letting δ and η tend to 0 gives us

$$\mathbb{E} \left[\exp \left(\frac{3\pi^2 (t \wedge \tau)}{D^2} \right) \sin \left(\frac{\pi \xi_{t \wedge \tau}}{D} \right) \right] \leq \sin \left(\frac{\pi |x - y|}{2D} \right).$$

Recall that $\xi_t = 0$ almost surely for $t \geq \tau$; thus we have

$$\mathbb{E} \left[\exp \left(\frac{3\pi^2 (t \wedge \tau)}{D^2} \right) \sin \left(\frac{\pi \xi_{t \wedge \tau}}{D} \right) \right] = \mathbb{E} \left[\exp \left(\frac{3\pi^2 t}{D^2} \right) \sin \left(\frac{\pi \xi_t}{D} \right) \right],$$

which leads to the desired result. \square

Now we are ready to prove

Theorem 3.3 (Fundamental gap conjecture). *If the potential function $V : \Omega \rightarrow \mathbb{R}$ of the Schrödinger operator $-\Delta + V$ is convex, then we have $\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}$.*

Proof. Applying Lemma 3.1 with $u_1(t, x) = e^{-\lambda_1 t} \phi_1(x)$, $(t, x) \in \mathbb{R}_+ \times \bar{\Omega}$, we see that $v = e^{-(\lambda_1 - \lambda_0)t} \frac{\phi_1}{\phi_0}$ is a smooth solution to (3.2). Set $v_0 = \frac{\phi_1}{\phi_0} \in C^1(\bar{\Omega}) \cap C^\infty(\Omega)$ for simplification of notations. The solution v has the probabilistic representation:

$$v(t, x) = \mathbb{E}v_0(X_t) \quad \text{and} \quad v(t, y) = \mathbb{E}v_0(Y_t),$$

where X_t and Y_t solve (2.7) and (2.8) respectively. Since the function v_0 is Lipschitz continuous on $\bar{\Omega}$ with a constant $K > 0$, we get

$$|v(t, x) - v(t, y)| \leq \mathbb{E}|v_0(X_t) - v_0(Y_t)| \leq K\mathbb{E}|X_t - Y_t| = 2K\mathbb{E}\xi_t.$$

Next it is easy to show that $\sin(\pi z/D) \geq 2z/D$ for $z \in [0, D/2]$, hence

$$|v(t, x) - v(t, y)| \leq KD\mathbb{E} \sin\left(\frac{\pi\xi_t}{D}\right) \leq KD \exp\left(-\frac{3\pi^2 t}{D^2}\right) \sin\left(\frac{\pi|x-y|}{2D}\right),$$

where the last inequality follows from Proposition 3.2. Substituting the expression of v into the above inequality leads to

$$e^{-(\lambda_1 - \lambda_0)t} |v_0(x) - v_0(y)| \leq KD \exp\left(-\frac{3\pi^2 t}{D^2}\right) \sin\left(\frac{\pi|x-y|}{2D}\right)$$

for all $x, y \in \Omega$ and $t \geq 0$. Since $v_0 = \frac{\phi_1}{\phi_0}$ is not constant, we conclude that $\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}$. \square

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